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Piecewise Monotone Interpolation

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In reaction to a recent paper by E. Passow in this Journal, it is shown that broken line interpolation as a scheme for piecewise monotone interpolation is hard to improve upon. It is also shown that a family of smooth piecewise polynomial interpolants, introduced by Swartz and Varga and noted by Passow to be piecewise monotone, converges monotonely, for fixed data, to a piecewise constant interpolant as the degree goes to infinity. Finally, piecewise monotone interpolation by splines with simple knots is discussed.

INTRODUCTION

This note is a reaction to the recent paper [4] on piecewise monotone spline interpolation by Eli Passow in this Journal. Passow considers piecewise monotone interpolation (PMI): Given a strictly increasing sequence $\mathbf{x} := (x_i)_0^k$ in the interval $I := [a, b]$ with $a = x_0$, $b = x_k$, a map P from $\mathbb{R}^{k+1}$ into the linear space $m(I)$ of all bounded real-valued functions on I is a PMI scheme (for $\mathbf{x}$) if

$$\begin{aligned} &\text{(i)} \quad (Py)(x_i) = y_i, && i = 0,\ldots, k, \\ \text{and}\quad &\text{(ii)} \quad Py \text{ is monotone on } [x_{i-1}, x_i], && i = 1,\ldots, k. \end{aligned} \tag{1}$$

Following Passow [4], we denote by $S_n^j = S_n^j(\mathbf{x})$ the linear subspace of $m(I)$ consisting of those f in $C^j(I)$ which, on each interval $[x_{i-1}, x_i]$, reduce to a

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polynomial of degree $\leq n$, $i = 1,\dots,k$. Passow pays special attention to the particular map H_n on $\mathbb{R}^{k+1}$ which is characterized by the conditions

$$\begin{aligned} &\text{(i)} \quad H_n y \in S^n_{2n+1}(\mathbf{x}), \\ &\text{(ii)} \quad (H_n y)(x_i) = y_i, \qquad i = 0,\dots,k, \\ &\text{(iii)} \quad (H_n y)^{(j)}(x_i) = 0, \qquad j = 1,\dots,n \quad \text{and} \quad i = 0,\dots,k. \end{aligned} \tag{2}$$

It should be said that this map was used earlier by Swartz and Varga [7] in their study of the error in spline interpolation and is, in that role, an application of an idea of Fejér [3] to *piecewise* polynomial interpolation. But Swartz and Varga [7] failed to comment on the piecewise monotonicity of $H_n y$ which is the point of Passow's paper. Passow [4] asserts the following

THEOREM. *For given $y \in \mathbb{R}^{k+1}$, $H_n y$ is the unique element of $S^n_{2n+1}(\mathbf{x})$ which takes on the value y_i at x_i, $i = 0,\dots,k$, and is monotone on each of the intervals $[x_{i-1}, x_i]$, $i = 1,\dots,k$.*

It is the purpose of this note to substantiate the following three comments regarding this theorem.

1. The uniqueness assertion in the theorem is certainly incorrect. But it is possible to show that any *linear* PMI scheme P into S^n_{2n+1} must agree with H_n on $[x_1, x_{k-1}]$.

2. With $\Delta x_i := x_{i+1} - x_i$, let $h := \max \Delta x_i$. If $y_i = f(x_i)$, all i, for some $f \in \mathbb{L}^{(2)}_\infty(I)$ (i.e., for some f with $f^{(1)}$ absolutely continuous and $f^{(2)}$ in $\mathbb{L}_\infty(I)$), then $H_0 y$ provides an $O(h^2)$-approximation to f while, for $n > 0$, $H_n y$ is only within $O(h)$ of f no matter how smooth f might be. In fact, $H_n y$ converges pointwise a.e. to a piecewise constant interpolant for f as $n \to \infty$. For this reason, we feel that H_n is not a particularly useful PMI scheme for $n > 0$ when compared with the simple scheme H_0 of broken line interpolation.

3. It might be objected that H_0 fails to provide a *smooth* p.m. (piecewise monotone) interpolant in contrast to H_n for $n > 0$. But if a p.m. spline interpolant in $C^n(I)$ is desired, then it is not necessary to use splines of degree $2n+1$. It is quite easy to construct p.m. spline interpolants with simple knots and of degree $n+1$ only.

1. UNIQUENESS OF A PMI SCHEME IN S^n_{2n+1}

Let P be a PMI scheme for $\mathbf{x}$, and let $e_i := (\delta_{ij})_{j=0}^k$. Then $\psi_i := Pe_i$ must vanish on each $[x_j, x_{j+1}]$ for $j \neq i-1, i$, by the monotonicity, and must be increasing on $[x_{i-1}, x_i]$, and decreasing on $[x_i, x_{i+1}]$ (take $x_{-1} := a$,

$x_{k+1} := b$). This shows that a linear map $P\colon \mathbb{R}^{k+1} \to m(I)$ is a PMI scheme if and only if $Py = \sum_{i=0}^{k} y_i \psi_i$, all $y \in \mathbb{R}^{k+1}$, for some $(\psi_i)_0^k$ in $m(I)$ with

$$\begin{aligned} &\text{(i)} \quad \text{support } \psi_i \subseteq (x_{i-1}, x_{i+1}),\ i = 0,\dots,k;\\ &\text{(ii)} \quad \text{on } [x_{i-1}, x_i],\ \psi_i \text{ is monotone increasing},\ i = 1,\dots,k,\\ &\text{(iii)} \quad \text{on } [x_{i-1}, x_i],\ \psi_{i-1} + \psi_i = 1,\ i = 1,\dots,k. \end{aligned} \tag{3}$$

In particular, a linear PMI scheme P is *local*. If also P maps into $C^n(I)$, then $\psi_i \in C^n$, all i, hence ψ_i vanishes $(n+1)$-fold at x_{i+1}, $i = 0,\dots,k-2$, and at x_{i-1}, $i = 2,\dots,k$. Therefore, then

$$(Py)^{(j)}(x_i) = 0, \qquad \text{for} \quad j = 1,\dots,n, \quad \text{and} \quad i = 1,\dots,k-1,$$

i.e., Py is flat of order n at each interior interpolation point. Thus, if P also maps into $S_{2n+1}^n(\mathbf{x})$, then, for $i = 2,\dots,k-1$, Py on $[x_{i-1}, x_i]$ must equal the *unique* polynomial of degree $\leqslant 2n+1$ which takes on the value y_j $(n+1)$-fold at x_j, $j = i-1, i$. Consequently, $Py = H_n y$ on $[x_1, x_{k-1}]$. It is in this sense only that, to quote [4], "there exists a unique $f \in S_{2n+1}^n(\mathbf{x})$ such that $f(x_i) = y_i$, $i = 0,\dots,k$ and f is monotone on each interval $[x_{i-1}, x_i]$, $i = 1,\dots,k$." Note that, even so, Py is *not* uniquely pinned down on $[x_0, x_1]$ or on $[x_{k-1}, x_k]$.

Of course, if we do not insist that P be linear, then there is no uniqueness whatsoever for $n > 0$, even if we insist that P map into $S_{2n+1}^n(\mathbf{x})$ (except on intervals $[x_{i-1}, x_i]$ for which $\Delta y_{i-1} = 0$). If, e.g., $n = 2$, and Δy_{i-1}, Δy_i are both positive for some i, then we can replace $H_n y$ on $[x_{i-1}, x_{i+1}]$ by any of the infinitely many piecewise cubics g which have a double knot at x_i, satisfy $g(x_j) = y_j$ for $j = i-1, i, i+1$, and have $g'(x_{i-1}) = g'(x_{i+1}) = 0$ and $0 \leqslant g'(x_i) \leqslant 3 \min\{\Delta y_{i-1}/\Delta x_{i-1}, \Delta y_i/\Delta x_i\}$.

Finally, we note that the localness of a linear interpolation map P: $\mathbb{R}^{k+1} \to m(I)$ follows already when P is only assumed to map monotone sequences to monotone functions. For, under these circumstances, $\phi_i := P(\sum_{j \geqslant i} e_j)$ is necessarily 0 to the left of $[x_{i-1}, x_i]$, and 1 to the right of that interval, and monotone increasing on that interval. But then, if P is linear, it must be of the form $Py = \sum_i y_i \psi_i$, with $\psi_i := \phi_i - \phi_{i+1}$, $i = 0,\dots,k$ (take $\phi_{k+1} := 0$) satisfying (3) as before.

2. Degree of Approximation

Consider now the question of how well Py approximates f in case $y = (f(x_i))_0^k$. If P is linear (the only case we consider here), then, by (3), $Py = f(x_i)\, \psi_i + f(x_{i+1})\, \psi_{i+1}$ with $\psi_i + \psi_{i+1} = 1$ and ψ_i, $\psi_{i+1} \geqslant 0$ on $[x_i, x_{i+1}]$. Therefore, for $x_i \leqslant x \leqslant x_{i+1}$,

$$\begin{aligned} |(f - Py)(x)| &\leqslant |f(x) - f(x_i)|\, \psi_i(x) + |f(x) - f(x_{i+1})|\, \psi_{i+1}(x)\\ &\leqslant \omega_f(\Delta x_i) \end{aligned} \tag{4}$$

with ω_f the modulus of continuity of f on I. It follows that $\|f - Py\|_I = O(h)$ with $h := \max \Delta x_i$ in case $f \in \mathbb{L}_\infty^{(1)}(I)$.

A faster rate of convergence is achieved essentially only by broken line interpolation. To make this precise, we restrict attention to the special case that

$$\psi_{i+1}(x) = \psi((x - x_i)/\Delta x_i) \qquad \text{for} \quad x_i \leqslant x \leqslant x_{i+1}, \quad i = 0,\ldots, k-1,$$

for some fixed function ψ defined on $[0, 1]$ and monotonely increasing there from $\psi(0) = 0$ to $\psi(1) = 1$. The PMI scheme H_n fits this description. We then have, for the specific function f given by $f(x) = x$, all x, and for $x_i \leqslant x \leqslant x_{i+1}$,

$$\begin{aligned} |f(x) - (Py)(x)| &= |x - x_i - (\Delta x_i)\,\psi((x - x_i)/\Delta x_i)| \\ &= \Delta x_i\,|(x - x_i)/\Delta x_i - \psi((x - x_i)/\Delta x_i)|. \end{aligned}$$

Hence, if $\|f - Py\|_I = o(h)$ for all sufficiently smooth f (e.g., for all analytic f) as $h \to 0$, then it follows that

$$(\Delta x_i/h)\,|(x - x_i)/\Delta x_i - \psi((x - x_i)/\Delta x_i)| \xrightarrow[h\to 0]{} 0, \quad \text{all} \quad x \in (x_i, x_{i+1}), \quad \text{all } i,$$

or, choosing i with $\Delta x_i = h$ and observing that $\{(x - x_i)/\Delta x_i : x_i < x < x_{i+1}\} = (0, 1)$,

$$\psi(t) = t, \qquad \text{all } t \in [0, 1].$$

Thus, H_n for $n > 0$ fails to provide $o(h)$-approximations to smooth f. This failure is particularly striking when $n \to \infty$. On each half interval $(x_j, x_{j+\frac{1}{2}})$, $j = 0, \frac{1}{2}, 1,\ldots, k - \frac{1}{2}$, $H_n y$ converges *monotonely* to the piecewise constant interpolant given by the rule

$$(H_\infty y)(x) := y_i \qquad \text{for} \quad x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}; \qquad (H_\infty y)(x_{i-\frac{1}{2}}) := y_{i-\frac{1}{2}}.$$

Here, $x_{i-\frac{1}{2}} := (x_{i-1} + x_i)/2$, all i. This fact is well known. To prove it, observe that the ψ for H_n is given by $\psi = \Psi_n$ with

$$\Psi_n(t) := A_n \int_0^t s^n(1 - s)^n\, ds$$

and

$$\begin{aligned} A_n &:= 1\Big/\int_0^1 s^n(1 - s)^n\, ds \\ &= (2n + 1)\binom{2n}{n} < (2n + 1)\sum_{i=0}^{2n}\binom{2n}{i} = (2n + 1)\,4^n. \end{aligned}$$

Therefore, on $[0, x]$ with $0 < x < \frac{1}{2}$,

$$0 \leqslant \Psi_n \leqslant (2n + 1)\,4^n x(x(1 - x))^n = (2n + 1)\, x \alpha_x^{\,n} \xrightarrow[n\to\infty]{} 0,$$

since $\alpha_x := 4x(1-x) < 1$. By symmetry, also $\Psi_n \to 1$ uniformly on $[1-x, 1]$ as $n \to \infty$. As to the monotonicity of convergence, observe that, on $(0, \frac{1}{2})$, $\Psi_n > \Psi_{n+1}$ since their difference $\Psi_n - \Psi_{n+1}$ is a polynomial of degree $2n+3$ which vanishes $(n+1)$-fold at 0 and at 1, and vanishes also at $\frac{1}{2}$, hence cannot vanish anywhere else, while its first nonzero derivative at 0 is positive.

3. A Linear PMI Scheme Using Smooth Splines

Consider now the possibility of a PMI scheme P with range in $S_{n+1}^n(\mathbf{z})$ for some strictly increasing $\mathbf{z}$ in I. Let $\phi_i := P(\sum_{j \geqslant i} e_j)$, $i = 0,..., k$, as at the end of Section 1. Then $\phi_i(x) = 0$ for $x \leqslant x_{i-1}$ and $\phi_i(x) = 1$ for $x \geqslant x_i$, hence ϕ_i' is a nontrivial spline of degree n with knot sequence $\mathbf{z}$ and with support in (x_{i-1}, x_i). Therefore (see, e.g., Curry and Schoenberg [2]) at least $n+2$ of the z_j's must lie in $[x_{i-1}, x_i]$ for $i = 1,..., k$.

Conversely, assume that, for $i = 1,..., k$, at least $n+2$ of the z_j's lie in $[x_{i-1}, x_i]$. This means that we can find $(r_i)_0^{k-1}$ such that $x_i \leqslant z_{r_i} < z_{r_i+n+1} \leqslant x_{i+1}$, all i. Define

$$\Phi_i(x) := \int_{-\infty}^{x} M_{r_i,n+1}(s)\, ds \tag{5}$$

with $M_{j,n+1}$ the B-spline of Curry and Schoenberg [2] of order $n+1$ with knots $z_j,..., z_{j+n+1}$. Then

$$M_{j,n+1}(x) = (n+1)[z_j,..., z_{j+n+1}](\cdot - x)_+^n$$

is a spline of degree n, with simple knots $z_j,..., z_{j+n+1}$, is positive on (z_j, z_{j+n+1}) and zero otherwise, and has unit integral. Therefore Φ_i, as defined in (5), is a spline of degree $n+1$ with simple knots $z_{r_i},..., z_{r_i+n+1}$ and rises strictly monotonely from a value of 0 at z_{r_i} (and to the left of z_{r_i}) to the value 1 at z_{r_i+n+1} (and to the right of z_{r_i+n+1}). But then,

$$P_{\mathbf{z},\mathbf{r}}y := y_0 + \sum_{i=0}^{k-1} (\Delta y_i)\, \Phi_i$$

defines a p.m. spline interpolant in $S_{n+1}^n(\mathbf{z})$.

Choose, in particular, $\mathbf{z}$ such that $z_{(n+1)i} = x_i$, $i = 0,..., k$. Then necessarily $r_i = (n+1)i$, all i. Denote the resulting $P_{\mathbf{z},\mathbf{r}}$ by $P_{\mathbf{z}}$. If we pick, in particular, $n = 2m$ and let $z_{(n+1)i+j}$ tend to x_i for $j = 0,..., m$, and to x_{i+1} for $j = m+1,..., n+1$, then, in the limit, $P_{\mathbf{z}} = H_m$. It may be noted parenthetically that Subbotin and Chernykh [5] defined a local interpolant of $f \in C^{(n)}(I)$ by elements of $S_{n+1}^n(\mathbf{z})$, where $\mathbf{z}$ is, essentially, a knot set such as we have just described. If, in their projection, we follow the idea of Fejér [3], of Subbotin [6], and of Swartz and Varga [7] by setting all occurrences of $f^{(j)}(x_i) = 0$, $1 \leqslant j \leqslant n$, then the resulting operator becomes P_z.

Incidentally, P_z obviously provides a *norm-reducing* interpolation scheme on $C(I)$ by splines of degree $n + 1$ which converges by (4) pointwise to the identity on $C(I)$ as $\max \Delta x_i \to 0$. This shows that splines are not inSAIN (see Chui *et al.* [1] for a somewhat different proof of this fact).

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